Journal of the Earth & Space Physics. Vol. 31, No. 1, 2005, P. 1-12

Step variable numerical orbit integration of a low earth orbiting satellite

Es-hagh, M*.

Islamic Azad University, Shahre-Rey Branch, Tehran, Iran

(Received: 11 Aug 2003 , Accepted: 31 May 2005)

Abstract

Investigation of the step variable numerical integration methods is the main purpose of this paper. Also it is shown how the one-dimensional numerical integration method of solving an ordinary differential equation can be employed to solve a three-dimensional differential equation numerically. In order to integrate the orbit, the step variable methods of Runge-Kutta-Fehlberg and Adams are employed and their accuracies are investigated.

Keywords: Integration, Orbit, Satellite, Adams, Fehlberg

1 INTRODUCTION

There are many numerical integration methods, which are used in orbit integration process. Generally, the numerical integration methods are classified into two groups: a. single step methods b. multi step methods. Both of these methods can be used as fixed step or variable step forms. In fixed step forms, the step size of integration is fixed and the integration process is carried out using this step size. However, in variable step methods, the step size of the integration depends on the accuracy of integration, i.e. it is selected in a way that the magnitude of local error be less than the accuracy of integration. If the local error is smaller than this accuracy then the initial step size is accepted, else the step size is rejected and is made smaller. This iteration in computing local error and reducing the size of integration interval is continued till the size of local error becomes below the acceptable limit. There are two methods for step variable numerical integration. These two methods could be employed as single step or multi step (predictor-corrector) methods; One of the most famous single step variable methods is the Runge-Kutta-Fehlberg method, (Babolian, 1994) and one of the well-known predictor-corrector step variable methods is the Adams-Bashforth and the Adams-Moulton step variable (Babolian, 1994) algorithm. Since the equation of motion of a satellite is a second order three-dimensional differential equation, it could be solved numerically using these methods. However these differential equations have to be converted to a system of first order differential equations. This system could also be solved numerically. In the next sections the aforementioned numerical methods will be presented in detail.

2 RUNGE-KUTTA-FEHLBERG INTEGRATION METHOD

The Runge-Kutta-Fehlberg (Babolian, 1994) integration method is similar to the ordinary Runge-Kutta approach and has been designed to solve the first order differential equations of the following form

$$\begin{aligned} y' &= f(t, y), \\ y(t_0) &= y_0. \end{aligned} \tag{1}$$

At first, the desired accuracy ε and initial step size h are selected, and then the following algorithmic solution is used (Babolian, 1994)

$$k1 = h\,f(t_n, y_n),$$
$$k2 = h\,f(t_n + \frac{1}{4}h,\ y_n + \frac{1}{4}k1),$$
$$k3 = h\,f(t_n + \frac{3}{8}h,\ y_n + \frac{3}{32}k1 + \frac{9}{32}k2),$$
$$k4 = h\,f(t_n + \frac{12}{13}h,\ y_n + \frac{1932}{2197}k1 + \frac{7200}{2197}k2 + \frac{7296}{2197}k3), \qquad (2)$$
$$k5 = h\,f(t_n + h,\ y_n + \frac{439}{216}k1 - 8\,k2 + \frac{3680}{513}k3 - \frac{845}{4104}k4),$$
$$k6 = h\,f(t_n + \frac{1}{2}h,\ y_n - \frac{8}{27}k1 + 2\,k2 - \frac{3544}{2565}k3 + \frac{1859}{4104}k4 - \frac{11}{40}k5).$$

The following relations should be employed to obtain the functional values in the next step

$$y_{n+1} = y_n + \frac{25}{216}k1 + \frac{1408}{2565}k3 + \frac{2197}{4104}k4 - \frac{1}{5}k5,$$
$$\tilde{y}_{n+1} = y_n + \frac{16}{135}k1 + \frac{6656}{12825}k3 + \frac{28561}{56430}k4 - \frac{9}{50}k5 + \frac{2}{55}k6. \qquad (3)$$

In order to check the initial step size the following criterion is defined as

$$\delta = 0.84\left(\frac{\varepsilon}{|r_{n+1}|}\right)^{1/4}, \qquad (4)$$

where

$$r_{n+1} = \frac{\tilde{y}_{n+1} - y_{n+1}}{h}, \qquad (5)$$

if $\delta \le 0.1$ then h is replaced by 0.1h, if $\delta \ge 4$ then h is replaced by 4h and if $1 < \delta < 4$ then h is changed to δh. Also if $|r_{n+1}| \le \varepsilon$ then the step size is accepted, consequently y_{n+1} is adopted as of the functional value in the next step, or else if $|r_{n+1}| > \varepsilon$ the step size is rejected and the algorithm should be iterated from equation (2).

According to the above algorithmic process, it is clear that if the local error is smaller than the desired accuracy, then the step size is accepted or else rejected. This algorithm has been designed to solve a first order differential equation. Therefore, in order to apply the method to the higher order differential equation, the differential equation should be converted to the system of first order differential equations. Then the system can be solved numerically by the Runge-Kutta-Fehlberg (RKF) method. According to Babolian (1994) the RKF method is subject to from instability.

3 ADAMS PREDICTOR-CORRECTOR STEP VARIABLE INTEGRATION METHOD

The Adams predictor-corrector step variable method is a multi step solution of differential equation. In order to start the algorithm of solution, some initial values, which are computed by the Runge-Kutta algorithm are required. The next functional value is predicted using the Adams-Bashforth predictor and corrected using the Adams-Moulton corrector. The step size must be controlled too. If the difference between predicted value and corrected value is 10 times smaller than the desired accuracy then the step size is accepted else it is rejected and changed by means of q factor. The following algorithm shows the simple Runge-Kutta algorithm in order to obtain the required initial values.

First, the initial step size h and the accuracy ε are selected; second, some initial values are computed by the Runge-Kutta method so that

$$k1 = h\,f(t_n, y_n),$$
$$k2 = h\,f(t_n + \frac{1}{2}h, y_n + \frac{1}{2}k1),$$
$$k3 = h\,f(t_n + \frac{1}{2}h, y_n + \frac{1}{2}k2), \qquad (6)$$
$$k4 = h\,f(t_n + h, y_n + k3),$$

and the functional values in next steps are computed recursively by

$$y_{n+1} = y_n + \frac{1}{6}(k1 + 2k2 + 2k3 + k4). \qquad (7)$$

The predicted value is obtained using Adams-Bashforth and corrected by the Adams-Moulton (Babolian, 1994) corrector as

$$y^p_{n+1} = y_n + \frac{h}{24}(55\, f(t_n, y_n) - 59 \times f(t_{n-1}, y_{n-1}) + 37 f(t_{n-2}, y_{n-2}) - 9 f(t_{n-3}, y_{n-3})), \qquad (8)$$

$$y^c_{n+1} = y_n + \frac{h}{24}(9 f(t_{n+1}, y^p_{n+1}) + 19 f(t_n, y_n) - 5 f(t_{n-1}, y_{n-1}) + f(t_{n-2}, y_{n-2})), \qquad (9)$$

the local error is computed as

$$\sigma = 0.1 \left| y^c_{n+1} - y^p_{n+1} \right|, \qquad (10)$$

also the q factor is computed

$$q = \sqrt[5]{\left(\frac{\varepsilon}{2\sigma}\right)}, \qquad (11)$$

if $0.1\varepsilon \le \sigma \le \varepsilon$ then the step size is accepted and the corrected value is a good approximation of functional value, if q>4 then h is replaced by 4h, otherwise it is changed to qh. If q<0.1 therefore h changes to 0.1h else to qh again. If $\sigma > \varepsilon$ then the step size is rejected and the algorithm must be iterated from equation (4) using the new step size. The above algorithm is designed for the solution of a first order differential equation.

4 SOLUTION OF EQUATION OF MOTION OF A SATELLITE

The equation of motion of a satellite is a second order vector differential equation, therefore it has to be converted to a system of first order differential equation in order to employ the above mentioned methods:

$$\ddot{\vec{r}} = -\frac{GM}{|\vec{r}|^3}\vec{r} + K = \begin{cases} \ddot{x} = -\frac{GM}{|\vec{r}|^3} x + K_x \\ \ddot{y} = -\frac{GM}{|\vec{r}|^3} y + K_y \\ \ddot{z} = -\frac{GM}{|\vec{r}|^3} z + K_z \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} v_x = \dot{x} \\ v_y = \dot{y} \\ v_z = \dot{z} \\ \dot{v}_x = -\frac{GM}{|\vec{r}|^3} x + K_x \\ \dot{v}_y = -\frac{GM}{|\vec{r}|^3} y + K_y \\ \dot{v}_z = -\frac{GM}{|\vec{r}|^3} z + K_z \end{cases} \qquad (12)$$

where, r is the position vector, GM is the product of gravitational constant and Earth's mass, k is the effects of all of the perturbing forces acting on a satellite. This system of first order differential equations could be solved by RKF or Adams step variable methods.

5 ORBIT INTEGRATION BY RUNGE-KUTTA-FEHLBERG METHOD

The mathematical description of this method was considered. Now in this section this method is employed in orbit integration. The solution of a system of first order differential equation can be obtained according to Babolian (1994). The following algorithm is very similar to the described algorithm of RKF but this algorithm includes many coefficients to be computed.

First, the initial step size h and the desired accuracy ε are selected, and then the following coefficients are computed

$$
\begin{aligned}
&k11 = h\dot{x}_n,\\
&k12 = h\dot{y}_n,\\
&k13 = h\dot{z}_n,\\
&k14 = hf_4(x_n, y_n, z_n),\\
&k15 = hf_5(x_n, y_n, z_n),\\
&k16 = hf_6(x_n, y_n, z_n),\\
&k21 = h(\dot{x}_n + (k14)/4),\\
&k22 = h(\dot{y}_n + (k15)/4),\\
&k23 = h(\dot{z}_n + (k16)/4),\\
&k24 = hf_4(x_n + (k11)/4,\\
&\qquad y_n + (k12)/4, z_n + (k13)/4),\\
&k25 = hf_5(x_n + (k11)/4,\\
&\qquad y_n + (k12)/4, z_n + (k13)/4),\\
&k26 = hf_6(x_n + (k11)/4,\\
&\qquad y_n + (k12)/4, z_n + (k13)/4),
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&k31 = h(\dot{x}_n + 3(k14)/32 + 9(k24)/32),\\
&k32 = h(\dot{y}_n + 3(k15)/32 + 9(k25)/32),\\
&k33 = h(\dot{z}_n + 3(k16)/32 + 9(k26)/32),\\
&k34 = hf_4(x_n + 3(k11)/32 + 9(k21)/32,\\
&\qquad y_n + 3(k12)/32 + 9(k22)/32,\\
&\qquad z_n + 3(k13)/32 + 9(k23)/32),\\
&k35 = hf_5(x_n + 3(k11)/32 + 9(k21)/32,\\
&\qquad y_n + 3(k12)/32 + 9(k22)/32,\\
&\qquad z_n + 3(k13)/32 + 9(k23)/32),\\
&k36 = hf_6(x_n + 3(k11)/32 + 9(k21)/32,\\
&\qquad y_n + 3(k12)/32 + 9(k22)/32,\\
&\qquad z_n + 3(k13)/32 + 9(k23)/32),
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&k41 = h(\dot{x}_n + 1932(k14)/2197 - 7200\times\\
&\qquad (k24)/2197 + 7296(k34)/2197),\\
&k42 = h(\dot{y}_n + 1932(k15)/2197 - 7200\times\\
&\qquad (k25)/2197 + 7296(k35)/2197),\\
&k43 = h(\dot{z}_n + 1932(k16)/2197 - 7200\times\\
&\qquad (k26)/2197 + 7296(k36)/2197),\\
&k44 = hf_4(x_n + 1932(k11)/2197 - 7200\times\\
&\qquad (k21)/2197 + 7296(k31)/2197, \cdots),\\
&k45 = hf_5(x_n + 1932(k11)/2197 - 7200\times\\
&\qquad (k21)/2197 + 7296(k31)/2197, \cdots),\\
&k46 = hf_6(x_n + 1932(k11)/2197 - 7200\times\\
&\qquad (k21)/2197 + 7296(k31)/2197, \cdots),
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
&k51 = h(\dot{x}_n + 439(k14)/216 - 8(k24) +\\
&\qquad 3680(k34)/513 - 845(k44)/4104),\\
&k52 = h(\dot{x}_n + 439(k15)/216 - 8(k25) +\\
&\qquad 3680(k35)/513 - 845(k45)/4104),\\
&k53 = h(\dot{x}_n + 439(k16)/216 - 8(k26) +\\
&\qquad 3680(k36)/513 - 845(k46)/4104),\\
&k54 = hf_4(x_n + 439(k11)/216 - 8(k21) +\\
&\qquad 3680(k31)/513 - 845(k41)/4104, \cdots),\\
&k55 = hf_5(x_n + 439(k11)/216 - 8(k21) +\\
&\qquad 3680(k31)/513 - 845(k41)/4104, \cdots),\\
&k56 = hf_6(x_n + 439(k11)/216 - 8(k21) +\\
&\qquad 3680(k31)/513 - 845(k41)/4104, \cdots),
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
&k61 = h(\dot{x}_n - 8(k14)/27 + 2(k24) - 3544(k34)/\\
&\qquad 2565 + 1859(k44)/4104 - 11(k54)/40),\\
&k62 = h(\dot{y}_n - 8(k15)/27 + 2(k25) - 3544(k35)/\\
&\qquad 2565 + 1859(k45)/4104 - 11(k55)/40),\\
&k63 = h(\dot{z}_n - 8(k16)/27 + 2(k26) - 3544(k36)/\\
&\qquad 2565 + 1859(k46)/4104 - 11(k56)/40),\\
&k64 = hf_4(x_n - 8(k11)/27 + 2(k21) - 3544(k31)/\\
&\qquad 2565 + 1859(k41)/4104 - 11(k51)/40, \cdots),\\
&k65 = hf_5(x_n - 8(k11)/27 + 2(k21) - 3544(k31)/\\
&\qquad 2565 + 1859(k41)/4104 - 11(k51)/40, \cdots),\\
&k66 = hf_6(x_n - 8(k11)/27 + 2(k21) - 3544(k31)/\\
&\qquad 2565 + 1859(k41)/4104 - 11(k51)/40, \cdots),
\end{aligned}
$$

$$
\begin{aligned}
&x_{n+1} = x_n + 25(k11)/216 + 1408(k31)/\\
&\qquad 2565 + 2197(k41)/4104 - (k51)/5,\\
&y_{n+1} = y_n + 25(k12)/216 + 1408(k32)/\\
&\qquad 2565 + 2197(k42)/4104 - (k52)/5,\\
&z_{n+1} = z_n + 25(k13)/216 + 1408(k33)/\\
&\qquad 2565 + 2197(k43)/4104 - (k53)/5,\\
&\dot{x}_{n+1} = \dot{x}_n + 25(k14)/216 + 1408(k34)/\\
&\qquad 2565 + 2197(k44)/4104 - (k54)/5,\\
&\dot{y}_{n+1} = \dot{y}_n + 25(k15)/216 + 1408(k35)/\\
&\qquad 2565 + 2197(k45)/4104 - (k55)/5,\\
&\dot{z}_{n+1} = \dot{z}_n + 25(k16)/216 + 1408(k36)/\\
&\qquad 2565 + 2197(k46)/4104 - (k56)/5,
\end{aligned}
\tag{17}
$$

$$\begin{aligned}
\tilde{x}_{n+1} &= x_n + 16(k11)/135 + 6656(k31)/\\
&\quad 12825 + 28561(k41)/56430 - 9\times\\
&\quad (k51)/50 + 2(k61)/55,\\
\tilde{y}_{n+1} &= y_n + 16(k12)/135 + 6656(k32)/\\
&\quad 12825 + 28561(k42)/56430 - 9\times\\
&\quad (k52)/50 + 2(k62)/55,\\
\tilde{z}_{n+1} &= z_n + 16(k13)/135 + 6656(k33)/\\
&\quad 12825 + 28561(k43)/56430 - 9\times\\
&\quad (k53)/50 + 2(k63)/55,\\
\tilde{\dot{x}}_{n+1} &= \dot{x}_n + 16(k14)/135 + 6656(k34)/\\
&\quad 12825 + 28561(k44)/56430 - 9\times\\
&\quad (k54)/50 + 2(k64)/55,\\
\tilde{\dot{y}}_{n+1} &= \dot{y}_n + 16(k15)/135 + 6656(k35)/\\
&\quad 12825 + 28561(k45)/56430 - 9\times\\
&\quad (k55)/50 + 2(k65)/55,\\
\tilde{\dot{z}}_{n+1} &= \dot{z}_n + 16(k16)/135 + 6656(k36)/\\
&\quad 12825 + 28561(k46)/56430 - 9\times\\
&\quad (k56)/50 + 2(k66)/55.
\end{aligned} \tag{18}$$

The state vector of position can be computed recursively using the following relations

$$\begin{aligned}
r_{x_{n+1}} &= \frac{\tilde{x}_{n+1} - x_{n+1}}{h},\\
r_{y_{n+1}} &= \frac{\tilde{y}_{n+1} - y_{n+1}}{h},\\
r_{z_{n+1}} &= \frac{\tilde{z}_{n+1} - z_{n+1}}{h},\\
r_{\dot{x}_{n+1}} &= \frac{\tilde{\dot{x}}_{n+1} - \dot{x}_{n+1}}{h},\\
r_{\dot{y}_{n+1}} &= \frac{\tilde{\dot{y}}_{n+1} - \dot{y}_{n+1}}{h},\\
r_{\dot{z}_{n+1}} &= \frac{\tilde{\dot{z}}_{n+1} - \dot{z}_{n+1}}{h},
\end{aligned} \tag{19}$$

where,

$$r_{n+1} = \sqrt{(r^2_{x_{n+1}} + r^2_{y_{n+1}} + r^2_{z_{n+1}} + r^2_{\dot{x}_{n+1}} + r^2_{\dot{y}_{n+1}} + r^2_{\dot{z}_{n+1}}}, \tag{20}$$

the following criterion is used to control the step size of integration in each step

$$\delta = 0.84\left(\frac{\varepsilon}{|r_{n+1}|}\right)^{1/4}, \tag{21}$$

if $\delta \le 0.1$ then h changes to 0.1h, if $\delta \ge 4$ then h is replaced by 4h and if $1 < \delta < 4$ then by δh. If $|r_{n+1}| \le \varepsilon$ then the step size is accepted and y_{n+1} is adopted as an approximation of the functional value at the next step, or else if $|r_{n+1}| > \varepsilon$ the step size is rejected then the algorithm should be iterated from equations of (11) until the desired accuracy is achieved.

6 ORBIT INTEGRATION BY ADAMS STEP VARIABLE INTEGRATION METHOD

The mathematical description of this method was considered before, now the system of first order differential equation should be solved using the Adams method too. The orbit integration algorithm of the Adams method will be presented as follows. As we mentioned before the orbit must be integrated by the Runge-Kutta method in order to provide some initial values for starting the Adams algorithm.

First, the initial step size h and accuracy ε are selected, and then the Runge-Kutta method is employed for computing the starter values,

$$\begin{aligned}
k11 &= h\dot{x}_n,\\
k12 &= h\dot{y}_n,\\
k13 &= h\dot{z}_n,\\
k14 &= hf_4(x_n, y_n, z_n),\\
k15 &= hf_5(x_n, y_n, z_n),\\
k16 &= hf_6(x_n, y_n, z_n),\\
k21 &= h(\dot{x}_n + (k14)/2),\\
k22 &= h(\dot{y}_n + (k15)/2),\\
k23 &= h(\dot{z}_n + (k16)/2),\\
k24 &= hf_4(x_n + (k11)/2, y_n + (k12)/2,\\
&\quad z_n + (k13)/2),\\
k25 &= hf_5(x_n + (k11)/2, y_n + (k12)/2,\\
&\quad z_n + (k13)/2),\\
k26 &= hf_6(x_n + (k11)/2, y_n + (k12)/2,\\
&\quad z_n + (k13)/2),
\end{aligned} \tag{22}$$

$$
\begin{aligned}
k31 &= h(\dot{x}_n + (k24)/2), \\
k32 &= h(\dot{y}_n + (k25)/2), \\
k33 &= h(\dot{z}_n + (k26)/2), \\
k34 &= hf_4(x_n + (k21)/2, y_n + (k22)/2, \\
&\quad z_n + (k23)/2), \\
k35 &= hf_5(x_n + (k21)/2, y_n + (k22)/2, \\
&\quad z_n + (k23)/2), \\
k36 &= hf_6(x_n + (k21)/2, y_n + (k22)/2, \\
&\quad z_n + (k23)/2),
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
k41 &= h(\dot{x}_n + k34), \\
k42 &= h(\dot{y}_n + k35), \\
k43 &= h(\dot{z}_n + k36), \\
k44 &= hf_4(x_n + k31, y_n + k32, z_n + \\
&\quad k33), \\
k45 &= hf_5(x_n + k31, y_n + k32, z_n + \\
&\quad k33), \\
k46 &= hf_6(x_n + k31, y_n + k32, z_n + \\
&\quad k33).
\end{aligned}
\tag{24}
$$

The state vector of position can be obtained recursively as

$$
\begin{aligned}
x_{n+1} &= x_n + (k11 + 2k21 + 2k31 + \\
&\quad k41)/6, \\
y_{n+1} &= y_n + (k12 + 2k22 + 2k32 + \\
&\quad k42)/6, \\
z_{n+1} &= z_n + (k13 + 2k23 + 2k33 + \\
&\quad k43)/6, \\
\dot{x}_{n+1} &= \dot{x}_n + (k14 + 2k24 + 2k34 + \\
&\quad k44)/6, \\
\dot{y}_{n+1} &= \dot{y}_n + (k15 + 2k25 + 2k35 + \\
&\quad k45)/6, \\
\dot{z}_{n+1} &= \dot{z}_n + (k16 + 2k26 + 2k36 + \\
&\quad k46)/6.
\end{aligned}
\tag{25}
$$

The predictor-corrector formulas are used for prediction and correction

$$
\begin{aligned}
k11 &= \dot{x}_{n+3}, \\
k12 &= \dot{y}_{n+3}, \\
k13 &= \dot{z}_{n+3}, \\
k14 &= f_4(x_{n+3}, y_{n+3}, z_{n+3}), \\
k15 &= f_5(x_{n+3}, y_{n+3}, z_{n+3}), \\
k16 &= f_6(x_{n+3}, y_{n+3}, z_{n+3}), \\
k21 &= \dot{x}_{n+2}, \\
k22 &= \dot{y}_{n+2}, \\
k23 &= \dot{z}_{n+2}, \\
k24 &= f_4(x_{n+2}, y_{n+2}, z_{n+2}), \\
k25 &= f_5(x_{n+2}, y_{n+2}, z_{n+2}), \\
k26 &= f_6(x_{n+2}, y_{n+2}, z_{n+2}),
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
k31 &= \dot{x}_{n+1}, \\
k32 &= \dot{y}_{n+1}, \\
k33 &= \dot{z}_{n+1}, \\
k34 &= f_4(x_{n+1}, y_{n+1}, z_{n+1}), \\
k35 &= f_5(x_{n+1}, y_{n+1}, z_{n+1}), \\
k36 &= f_6(x_{n+1}, y_{n+1}, z_{n+1}), \\
k41 &= \dot{x}_n, \\
k42 &= \dot{y}_n, \\
k43 &= \dot{z}_n, \\
k44 &= f_4(x_n, y_n, z_n), \\
k45 &= f_5(x_n, y_n, z_n), \\
k46 &= f_6(x_n, y_n, z_n),
\end{aligned}
\tag{27}
$$

The Adams-Bashforth predictors are:

$$
\begin{aligned}
x^p_{n+4} &= x_{n+3} + h(55k11 - 59k21 + \\
&\quad 37k31 - 9k41)/24, \\
y^p_{n+4} &= y_{n+3} + h(55k12 - 59k22 + \\
&\quad 37k32 - 9k42)/24, \\
z^p_{n+4} &= z_{n+3} + h(55k13 - 59k23 + \\
&\quad 37k33 - 9k43)/24, \\
\dot{x}^p_{n+4} &= \dot{x}_{n+3} + h(55k14 - 59k24 + \\
&\quad 37k34 - 9k44)/24, \\
\dot{y}^p_{n+4} &= \dot{y}_{n+3} + h(55k15 - 59k25 + \\
&\quad 37k35 - 9k45)/24, \\
\dot{z}^p_{n+4} &= \dot{z}_{n+3} + h(55k16 - 59k26 + \\
&\quad 37k36 - 9k46)/24.
\end{aligned}
\tag{28}
$$

Now these predicted values are corrected by the Adams-Moulton method

$$
\begin{aligned}
x^c_{n+4} &= x_{n+3} + h(9(\dot{x}^p_{n+1}) + 19k11 - \\
&\quad 5k21 - k31)/24, \\
y^c_{n+4} &= y_{n+3} + h(9(\dot{y}^p_{n+1}) + 19k12 - \\
&\quad 5k22 - k32)/24, \\
z^c_{n+4} &= z_{n+3} + h(9(\dot{z}^p_{n+1}) + 19k13 - \\
&\quad 5k23 - k33)/24, \\
\dot{x}^c_{n+4} &= \dot{x}_{n+3} + h(9f_4(x^p_{n+4}, y^p_{n+4}, z^p_{n+4}) + \\
&\quad 19k14 - 5k24 - k34)/24, \\
\dot{y}^c_{n+4} &= \dot{y}_{n+3} + h(9f_5(x^p_{n+4}, y^p_{n+4}, z^p_{n+4}) + \\
&\quad 19k15 - 5k25 - k35)/24, \\
\dot{z}^c_{n+4} &= \dot{z}_{n+3} + h(9f_6(x^p_{n+4}, y^p_{n+4}, z^p_{n+4}) + \\
&\quad 19k16 - 5k26 - k36)/24.
\end{aligned}
\tag{29}
$$

The difference between predicted value and the corrected value is computed

$$
\begin{aligned}
\sigma_{x_{n+1}} &= 0.1\left|x^c_{n+1} - x^p_{n+1}\right|, \\
\sigma_{y_{n+1}} &= 0.1\left|y^c_{n+1} - y^p_{n+1}\right|, \\
\sigma_{z_{n+1}} &= 0.1\left|z^c_{n+1} - z^p_{n+1}\right|, \\
\sigma_{x_{n+1}} &= 0.1\left|\dot{x}^c_{n+1} - \dot{x}^p_{n+1}\right|, \\
\sigma_{y_{n+1}} &= 0.1\left|\dot{y}^c_{n+1} - \dot{y}^p_{n+1}\right|, \\
\sigma_{\dot{z}_{n+1}} &= 0.1\left|\dot{z}^c_{n+1} - \dot{z}^p_{n+1}\right|,
\end{aligned}
\tag{30}
$$

and

$$
\sigma = \sqrt{(\sigma^2_{x_{n+1}} + \sigma^2_{y_{n+1}} + \sigma^2_{z_{n+1}} + \sigma^2_{\dot{x}_{n+1}} + \sigma^2_{\dot{y}_{n+1}} + \sigma^2_{\dot{z}_{n+1}}}, \tag{31}
$$

if the difference is 10 times smaller than the desired accuracy, then the step size is accepted. After that the q factor is computed as

$$
q = \sqrt[5]{\left(\frac{\varepsilon}{2\sigma}\right)}, \tag{32}
$$

and if $0.1\varepsilon \le \sigma \le \varepsilon$ then the step size is accepted and the corrected value is a good approximation to the functional value. If q>4 then h is replaced by 4h, otherwise it is replaced by qh , if q<0.1 then h changes to 0.1h else by qh. If $\sigma > \varepsilon$ then the step size is rejected. Also the algorithm has to be iterated from equation (20) using new step size.

7 NUMERICAL RESULTS

According to Kepler's laws the satellite's trajectory is an ellipse in a central field. This trajectory is also called a theoretical orbit. In order to investigate the global error of integration the integrated orbit in a central gravitational field is compared to the theoretical orbit. The RKF method is a step variable method and its integration step size depends on the local error of integration. The following figures show the step size and local error for a low Earth orbiting satellite in one revolution, the period of this revolution is about the 5600 second,

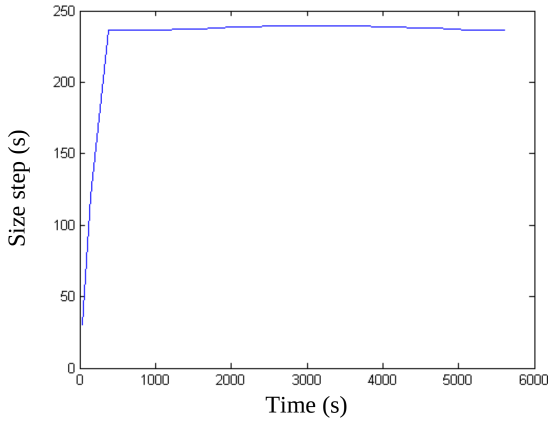

Figure 1. Step size with local error 0.1 m.

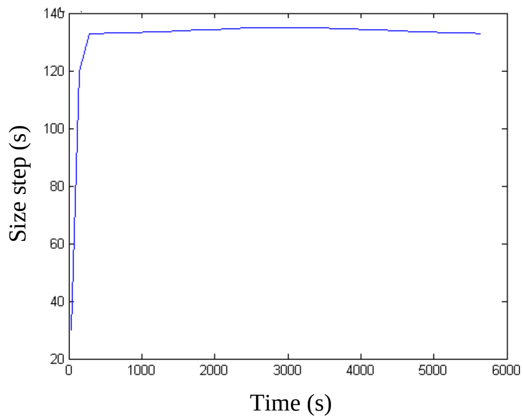

Figure 2. Step size with local error 0.01 m.

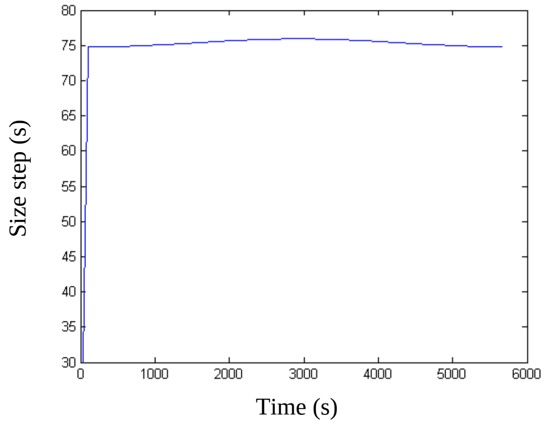

Figure 3. Step size with local error 0.001 m.

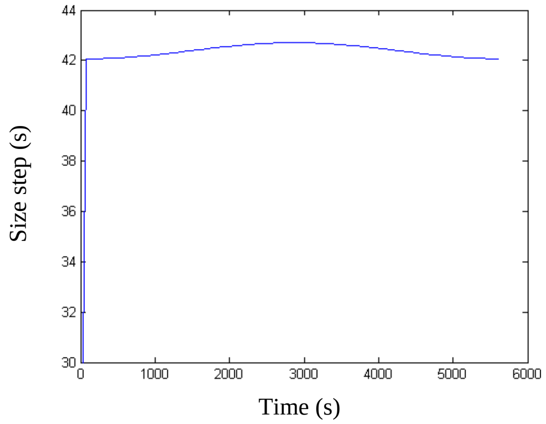

Figure 4. Step size with local error 0.0001 m.

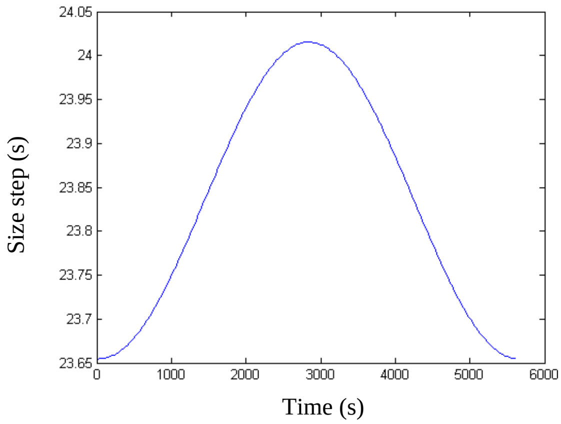

Figure 5. Step size with local error 0.00001 m.

The above figures show the dependence of the step size of integration and period of revolution of the satellite at an altitude of about 400 Km. Figure 1 shows that the initial step size selected for starting the process changes abruptly from 30 second to 240 second with local error of about 0.1 meter and the integration will be continued using this step size. Figure 2 presents the variations of step size with local error of 0.01 meter. According to this figure one can see that the step size changes suddenly from 30 to about 130 second in one revolution of the satellite. Figure 3 shows the relation between step size and local error of 0.001 meter, it can be seen that the step size changes to 75 second from 30 second. Also by referring to the figure one can see that the step size changes to 42 second with local error of 0.0001 meter. Finally, figure 5 shows the step size variations due to the local error of about 0.00001 meter. It is understood that the step size varies between 23.5 -24 second. It is clear that there are no intensive variations in this situation.

The difference between theoretical orbit and integrated orbit using these step sizes is as follows:

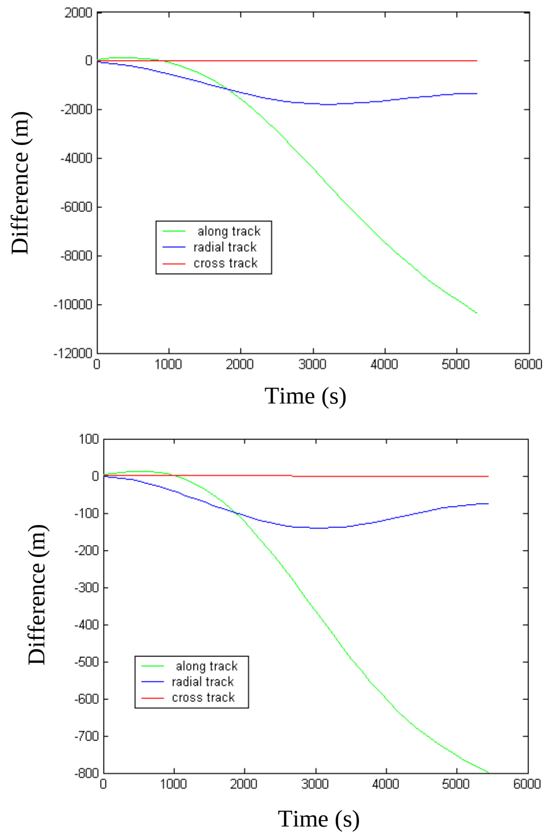

Figure 6. Difference between theoretical orbit and integrated orbit with local error of 0.1 and 0.01 m.

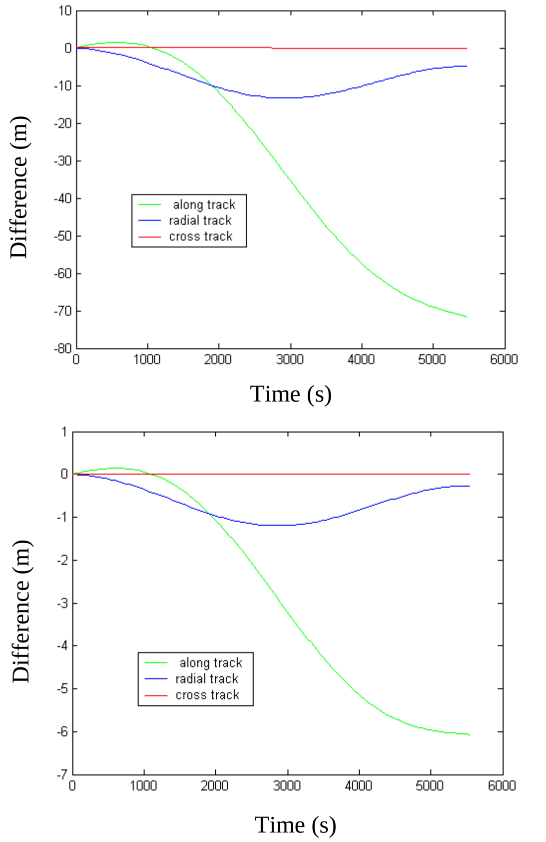

Figure 7. Difference between theoretical orbit and integrated orbit with local error of 0.001 and 0.0001 m.

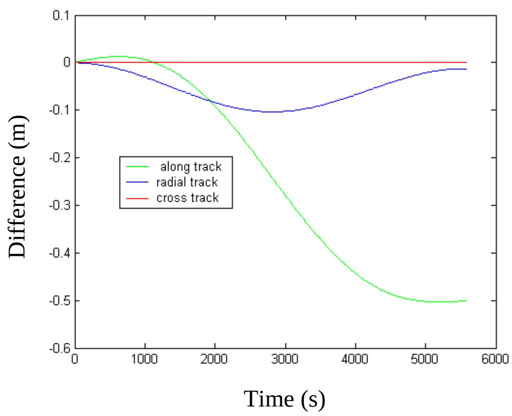

Figure 8. Difference between theoretical orbit and integrated orbit with local error of 0.00001 m.

According to figure 6 to figure 8, one can understand that the difference between the theoretical orbit and integrated orbit depends exactly on the local error. This figures show the same behavior of this difference but with a difference in magnitude of the errors. It is evident that the error of numerical integration method (local error) has to be smaller than the desired accuracy of the orbit. Figure 1 shows that the local error of about 0.1 meters provides a huge difference between theoretical orbit and integrated orbit, which means that the local error of 0.1 meter is not acceptable for our purposes at all. According to the figures one can conclude that by reducing the local errors the difference between theoretical and integrated orbit is also reduced. Also one can say that the local error of 0.00001 meter is permissible to compute the satellite orbit and obtain the suitable step size for the integration process.

The following figures show the variations of the step size with different local errors in the Adams step variable method.

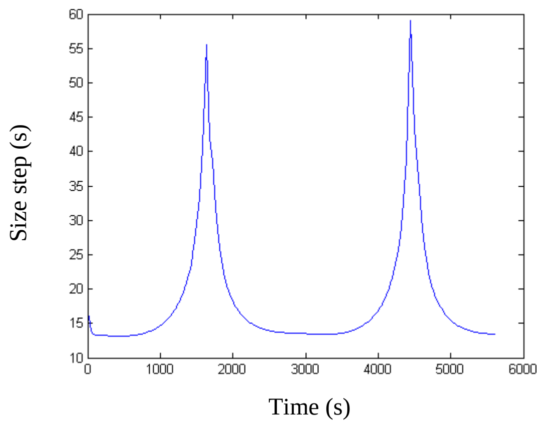

Figure 9. Step size with local error 0.01 m.

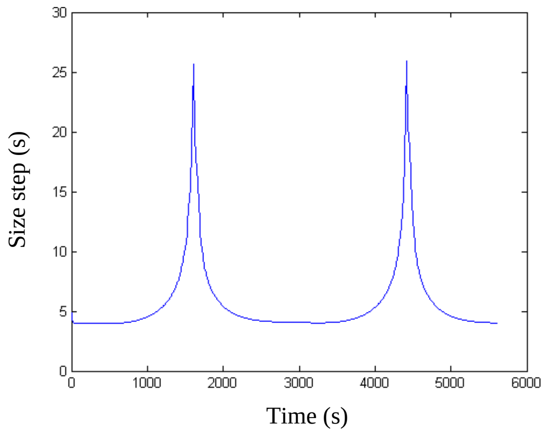

Figure 10. Step size with local error 0.001 m.

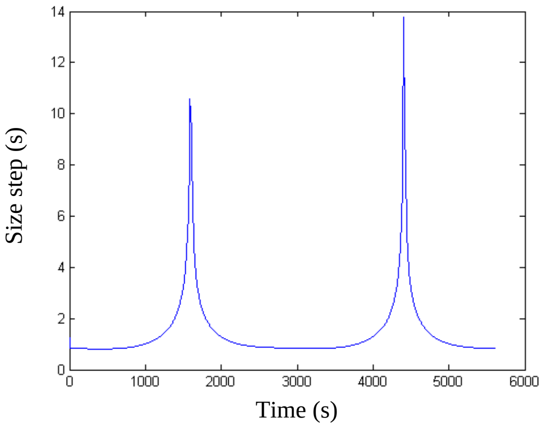

Figure 11. Step size with local error 0.001 m.

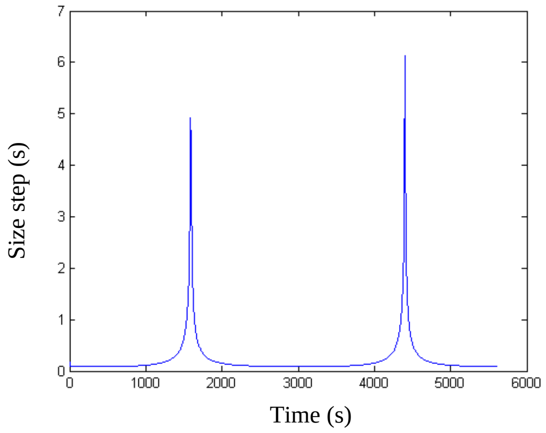

Figure 12. Step size with local error 0.0001 m.

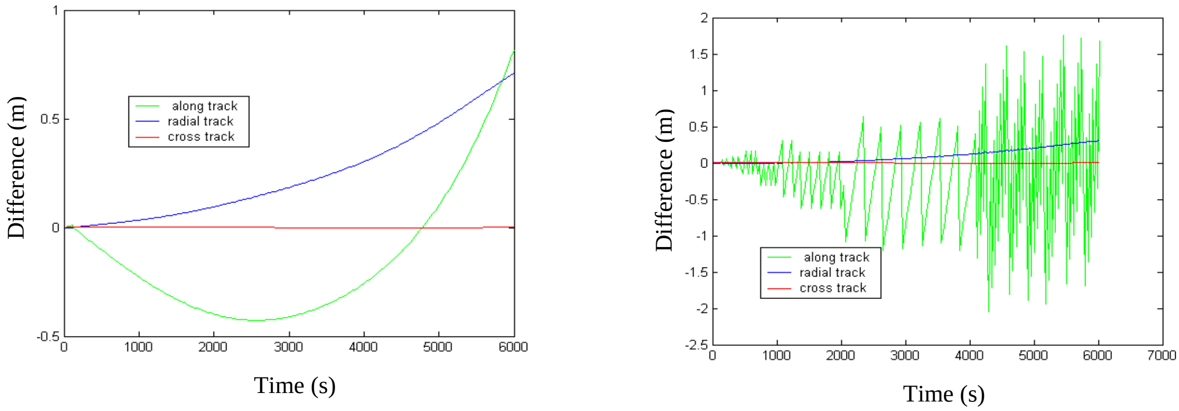

Figure 13. Difference between theoretical orbit and integrated orbit by Adams method with accuracy of 0.1 m and 0.01 m.

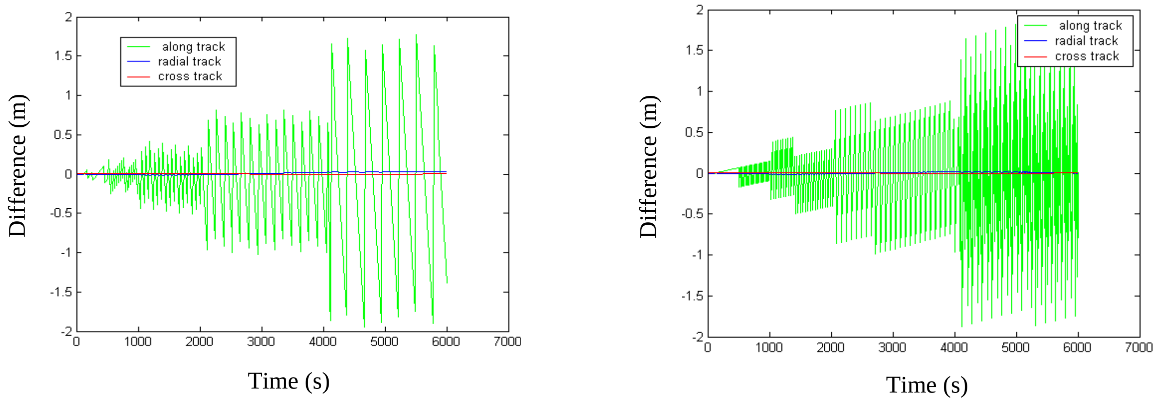

Figure 14. Difference between theoretical orbit and integrated orbit by Adams method with accuracy of 0.001 m and 0.00001 m.

Figure 9 shows the step size variations due to the local error of 0.01 meter. It can be seen that there are two sharp variations during one revolution of the satellite. The average of the step size variations is about 17 second reduced from 30 second. Figure 10 presents this step size average reduced to less than 5 second due to the 0.001 meter local error. According to the figures it is concluded that the step sizes of integration have the same behavior but differ with magnitude. The figures show the stable step sizes except two points during one revolution.

The difference between theoretical orbit and integrated orbit using the Adams step variable method is also investigated and the results are as follows:

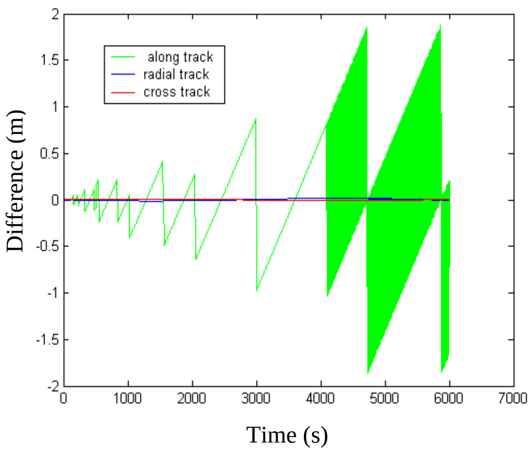

Figure 15. Difference between theoretical orbit and integrated orbit by Adams method with accuracy of 0.00001 m.

The above figures show the differences between theoretical orbit and integrated orbit using the Adams step variable method. Figure 13 shows that the different changes between -0.5-1 m. in 3 directions of along, cross and radial track components due to the local error of 0.1 m. Figure 14 and figure 15 show that the difference has random behavior in the along track component and the variations in other components have been reduced, i. e., the geometry of the orbit is approximately preserved. Also according to the figures one can conclude that there is no improvement in the difference between theoretical and integrated orbit by reducing the local error of integration during one revolution.

8 CONCLUSIONS AND RECOMMENDATIONS

According to the numerical results presented in the previous sections, it is clear that the Runge-Kutta-Fehlberg method is a step size dependent method. Also this method shows that it provides the worst results in large step sizes but gives the most satisfactory results in small step sizes and local error. In this method the step size is changed during of integration process and it does not have a fixed value. The Runge-Kutta-Fehlberg algorithm has large variations in large local errors, but it can be improved using small local error and there is not large variation in step sizes. Clearly, the difference between the theoretical and integrated orbit have to be reduced too. The Runge-Kutta-Fehlberg method changes the geometry of the orbit of the satellite but the Adams method is more stable than the RKF method and regardless of the accuracy, the geometry of satellite's orbit is preserved, i.e., it does not change the orbit geometry. The numerical values of the results of the Adams method are very close to each other, therefore, this method is more stable than the RKF method but in small step sizes the RKF method is suitable and stable for orbit integration. Also the Adams method is not economical in order to be integrated. For the local error of about 0.1 m the RKF method gives the step size of about 240 second while the Adams method gives the step size of about 17 second does this belong to the previous or next seuteuce. With Adams method the step size of about 6 second is obtained but the RKF method yields the step size of about 23 second and it is more economical than the Adams method. Therefore, the Adams method could be employed in large step size because of its stability too, but the RKF method is stable in small step size. Since the Adams step variable method is very stable in any step size therefore, it is recommended that the Adams method be used for long arc orbit integration or in low resolutions (high step size) orbit integration and in contrast it is recommended that for high resolution (low step size) the RKF method be used.

REFERENCES

Babolian, E., 1994, Numerical methods Elm va Sanat University Publication, Department of mathematics, Tehran, Iran (in Farsi).

Binsnath. S., and Langley R. B., 2001, Precise orbit determination of low earth orbiter with GPS point positioning, University of Newbrunswick.

Buffet. A., 1985, Short arc orbit improvement of GPS satellite Technical report, University of Calgary.

Es-hagh, M., 2003, Precise orbit determination of a low Earth orbiting satellite M.Sc. thesis, K. N. Toosi University of Technology, Faculty of Geodesy and Geomatic Engineering, Department of Geodesy.

Hofman, B., Lichteneffer, H., and Collins, J., 1992, Global positioning system Springer-Verlag, New York.

McCarthy, D., 1996, IERS Technical note 21 US Naval observatory, Central Bureau of IERS-Observatoire de Paris.

Parrot, D., 1989, Short arc orbit improvement for GPS satellites Department of surveying Engineering University of New Brunswick.

Santos, M. C., 1994, On real time orbit improvement for GPS satellites. The University of New Brunswick, department of Geodesy and Geomatic Engineering.

Scherneck, H. G., 1996, Explanatory supplement: Ocean loading tide, Onsala Space Observatory Chalmers University of Technology of Sweden.

Shidfar. A., 1994, Differential equations, University of Elm va Sanat, Tehran (in Farsi).

Su, H., 2000, Orbit determination of IGSO, GEO and MEO satellites, Ph.D thesis on University of Bundeswehr Munchen.

Vanicek, P., and Krakiwsky, E., 1986, Geodesy the concepts University of New Brunswick of Canada, Elsevier, Amsterdam.

Wolf, R., 2000, Satellite orbit and ephemeris Determination using inter satellite links Ph.D Thesis on University of Bundeswehr Munchen.